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HOMOGENEOUS POLYNOMIALS ON THE BALL AND POLYNOMIAL BASES

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ABSTRACT
It is shown that the spaces of homogeneous polynomials on the complex 2-ball are uniformly isomorphic to l^∞-spaces. The argument is based on explicit constructions and the decomposition method. A new construction is given of bases in the space A_N of monomials $1, z, z^2, \dots, z^{N-1}$ on the disc (due to Bochkarev [Boc]). Also using decomposition methods, the existence of a base in the ball algebra is obtained.

1. Introduction

Denote $B_2 = \{(z, w) \in \mathbf{C}^2 \mid |z|^2 + |w|^2 \leqq 1\}$ the complex ball and $S_2 = \{\zeta \in B_2 \mid |\zeta| \equiv \langle \zeta, \zeta \rangle^{1/2} = 1\}$. Let σ denote the invariant measure on S_2. For $N = 1, 2, \dots$, consider the space

$$W_N = [z^k w^{N-k} \mid 0 \leqq k \leqq N]$$

of homogeneous polynomials of degree N. The orthogonal projection on W_N is given by

$$P_N f(\zeta) = \frac{\int f(\eta) \langle \zeta, \eta \rangle^N \sigma(d\eta)}{\int |\eta_1|^{2N} \sigma(d\eta)}. \tag{1.1}$$

Under parametrization $z = \sqrt{\rho}\, e^{2\pi i \theta}$, $w = \sqrt{1-\rho}\, e^{2\pi i \psi}$ $(0 \leqq \rho \leqq 1,\ 0 \leqq \theta, \psi \leqq 1)$, $d\sigma = d\rho\, d\theta\, d\psi$ (cf. [Ru]). The operators P_N have the remarkable property of being uniformly bounded on $L^1(S_2)$ and $L^\infty(S_2)$. Indeed

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$$\| P_N \| = \frac{\int |\eta_1|^N \sigma(d\eta)}{\int |\eta_1|^{2N} \sigma(d\eta)} < 2. \tag{1.2}$$

This fact was observed in [R–W]. The result remains valid in general (d variables) but the discussion in this paper is limited to the case $d = 2$. Our purpose is to give an affirmative solution to the problem considered in [R–W], namely the validity of a uniform distance estimate

$$d(W_N^\infty, l_{N+1}^\infty) \leqq c. \tag{1.3}$$

Here W_N^∞ stands for the space W_N equipped with uniform norm (i.e. as subspace of $L^\infty(S_2)$) and $d(\cdot,\cdot)$ is the usual Banach–Mazur distance between (finite dimensional) normed spaces.

The spaces W_N^∞ are P_λ-spaces for $\lambda = 2$, i.e. they appear as complemented subspaces of L^∞ with projection constant $\leqq \lambda$. It is an open problem whether there is a function $\varphi(\lambda)$ such that

$$d(X, l_{\dim X}^\infty) \leqq \varphi(\lambda) \tag{1.4}$$

whenever X is a finite dimensional P_λ-space [Rut].

The correctness of (1.4) was verified by M. Zippin [Z] in the "almost-isometric" case, i.e. for $\lambda < 1 + \varepsilon$ (ε = absolute). Moreover $\lim_{\lambda \to 1} \varphi(\lambda) = 1$.

Essentially speaking, the spaces W_N^∞ (and their several variable analogues) are the only known examples of non-trivial P_λ's. During recent years, there has been a certain effort expended to understand their structure. D. Kazhdan [K] expressed doubts about the existence of irreducible representations of the unitary group on l^∞-spaces. This and observations made in [R–W] made the homogeneous polynomial spaces natural candidates for a counterexample to the "finite-dimensional" P_λ-problem. In this paper I will disprove this belief by showing

THEOREM. *There is an absolute constant C (independent of N) such that $d(W_N^\infty, l_{N+1}^\infty) \leqq C$.*

I do not intend to investigate the case of several variables here. The previous result is of interest for various reasons. From a classical analysis point of view, it elucidates the nature of a rather important function space, while its meaning to Banach space theory is the support to a well-known conjecture. The proof of the theorem will not provide an explicit l^∞-basis. The argument used a version of the "finite decomposition method" in the spirit of [B–D–G–J–N] or [J–S] for instance. The drawback of this method is that the isomorphism is not explicit.

The first step in the proof is the replacement of W_N^∞ by a "weighted norm" space on the linear span of characters

$$1, e^{2\pi i\theta}, \ldots, e^{2\pi iN\theta}.$$

This alternative description was pointed out in [R–W] and has been used in [B_1]. As I observed recently, it is possible to prove directly the P_λ-property from this alternative description, ignoring the origin of the space. In fact, the space on characters appears as a member of a larger class of P_λ-spaces all of which, eventually, turn out to be l^∞-spaces.

As mentioned above, the isomorphism with l^∞ is shown by means of a finite dimension decomposition method. In my original approach, I used the work of S. V. Bochkarev [Boc] on bases in the space $A_n = [e^{2\pi ik\theta} \mid 0 \leqq k < n]$, i.e. the linear span of the first n characters on the circle Π equipped with the uniform norm. Piecing together the spaces associated to the Bochkarev basis over a decomposition of the interval $[0, 1, \ldots, N]$ in certain subintervals corresponding to the weighted norm description of W_N^∞ (each of the subintervals is identified with A_n for appropriate n) leads to a monotone system of P_λ-subspaces of W_N^∞. Here λ is some uniform constant. These intermediate spaces are the basic ingredient when using the decomposition method.

It is of importance for our purpose to have a good understanding of the Lebesgue functions associated to the Bochkarev basis. Their behaviour may be figured out from the construction but is not explicitly stated in [Boc]. Subsequently, I came up with an alternative construction of a basis in A_n, mainly motivated by the need for using it in the context of W_N^∞. For this basis, the Lebesgue functions are almost immediate and are essentially multipliers. This basis is explicit and particularly simple to describe. It also presents the other features of the Bochkarev basis (unconditionality in L^1, etc.). They are however irrelevant for our purpose here. Throughout the paper, c will denote constants.

2. Equivalent description of the norm

The description used here is formally different from the one pointed out in [R–W] (in our discussion, equivalence means up to a constant). If $f = \Sigma_{0 \leqq k \leqq N} a_k z^k w^{N-k}$, then using the above parametrization,

$$\| f \|_{L^\infty(S_2)} = \sup_{\substack{0 \leqq \rho \leqq 1 \\ 0 \leqq \theta \leqq 1}} \left| \sum_0^N a_k \rho^{k/2} (1-\rho)^{(N-k)/2} e^{2\pi i k\theta} \right| . \tag{2.1}$$

The expression $\rho^{k/2}(1-\rho)^{(N-k)/2}$ will be largest for $\rho = k/N$ and one verifies that

$$\begin{aligned} &\rho^{k/2}(1-\rho)^{(N-k)/2} \\ &\leqq \left(\frac{k}{N}\right)^{k/2} \left(\frac{N-k}{N}\right)^{(N-k)/2} \exp\left\{ -c \frac{(N\rho - k)^2}{\min(N\rho + k, N(2-\rho) - k)} \right\} . \end{aligned} \tag{2.2}$$

Observe in particular that for given ρ, $\rho \sim k_0/N$, $0 \leqq k_0 \leqq N/2$ say,

$$\rho^{k/2}(1-\rho)^{(N-k)/2} \sim \left(\frac{k}{N}\right)^{k/2} \left(\frac{N-k}{N}\right)^{(N-k)/2} \quad \text{for } |k - k_0| \sim k_0^{1/2}.$$

Consider the diagonal operator between W_N and $[1, e^{2\pi i\theta}, \dots, e^{2\pi i N\theta}]$ given by

$$Df = \sum a_k \left(\frac{k}{N}\right)^{k/2} \left(\frac{N-k}{N}\right)^{(N-k)/2} e^{2\pi i k\theta}. \tag{2.3}$$

From (2.2) and elementary considerations (cf. [B$_1$]), it turns out that

$$\| f \|_{L^\infty(S_2)} \sim \| Df \|_A \tag{2.4}$$

where $\| g \|_A = \sup \| g * K \|_\infty$, the supremum being taken over "admissible" K. A kernel K is called admissible, provided

$$\begin{aligned} &\| K \|_1 = \| K \|_{L^1(\Pi)} \leqq 1, \\ &\operatorname{supp} \hat{K} \subset [a, b] \subset [0, N] \end{aligned} \tag{2.5}$$

where

$$b - a \leqq \min(a^{1/2}, (N-b)^{1/2}) + 1. \tag{2.6}$$

Thus D gives an isomorphism between W_N^∞ and $[1, \dots, e^{2\pi i N\theta}]_A$ (the constants involved are of course independent of N). Define

$$X = X_N = [1, e^{2\pi i\theta}, \dots, e^{2\pi i N\theta}]_A.$$

The theorem stated in the introduction may then be reformulated as

$$d(X_N, l_{N+1}^\infty) < c. \tag{2.7}$$

Observe that if $\{K_\alpha\}$ is chosen such that

(2.8) K_α is admissible for each α,

(2.9) $\sum \hat{K}_\alpha = 1$ on $[0, N]$,

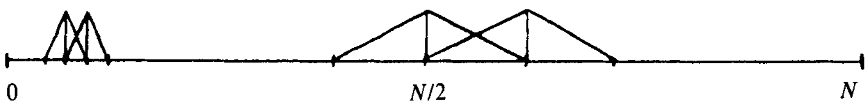
(2.10) supp $\hat{K}_\alpha$ have bounded overlap,

(2.11) $\#\{\alpha \mid \operatorname{supp} \hat{K}_\alpha \cap [a, b] \neq \varnothing\} < c$,

for any interval $[a, b]$ satisfying (2.6), then one already has

(2.12) $$\| g \|_A \sim \sup_\alpha \| g * K_\alpha \|_\infty.$$

A system of such K_α is easily constructed, letting $\hat{K}_\alpha$ be triangular based on suitable intervals



(Each K_α is the product of a Féjèr-kernel and some character.) Our next purpose is to show directly from the definition of X_N the P_λ-property. This argument will be of importance later on. It will also show that there are many variants of (2.6) when defining the admissible kernels, leading to different norms on $[1, e^{2\pi i\theta}, \ldots, e^{2\pi i N\theta}]$ and P_λ-spaces.

Lemma 2.13. *Consider V where $\hat{V}$ is given by*

$$\begin{cases} \hat{V} = 1 \text{ on } [0, n], \\ \hat{V} = 0 \text{ outside } [-m + 1, n + m - 1], \\ \hat{V} \text{ piecewise linear (trapezoïdal)}, \end{cases}$$

and denote for $j = 0, 1, \ldots, n + m - 1$

$$V_j(\theta) = V\left(\theta - \frac{j}{n + m}\right).$$

Then, assuming $m < n$,

(2.14) $e^{2\pi i k\theta}$ *belongs to* $[V_j]$ *for* $0 \leqq k \leqq n$

and also

$$V_j\left(\frac{j'}{n+m}\right)=(n+m)\delta_{jj'}. \tag{2.15}$$

PROOF. To see (2.14), write for $0\leqq k\leqq n$

$$e^{2\pi ik\theta}=\frac{1}{n+m}\sum_0^{n+m-1}e^{-2\pi ikj/(n+m)}V_j. \tag{2.16}$$

To get (2.15), write

$$V\left(\frac{j}{n+m}\right)=\sum_0^n e^{2\pi ijk/(n+m)}+\sum_{-m+1}^{-1}\frac{m+k}{m}e^{2\pi ijk/(n+m)}$$
$$+\sum_{n+1}^{n+m-1}\frac{m+n-k}{m}e^{2\pi ijk/(n+m)}$$

and put in the second sum $k'=k+m+n$.

LEMMA 2.17. *Under the hypothesis of Lemma* 2.13

$$\left\|\sum_0^{n+m-1}a_jV_j\right\|_\infty\leqq c\frac{n^2}{m}\max|a_j|. \tag{2.18}$$

PROOF. Use the decay estimate $|V_j(x)|\leqq cn[1+m^2\sin^2\pi x]^{-1}$.

Choosing some $m<n$, $m\sim n$, it follows from Lemmas 2.13 and 2.17 that $[V_j\mid 0\leqq j<n+m]$ is an l^∞_{m+n} isomorph (the basis vectors are given by $(1/(n+m))V_j$) containing $[1,\dots,e^{2\pi in\theta}]$. Moreover

$$[V_j]\subset[e^{2\pi ik\theta}\mid -m<k<n+m].$$

We now return to the space X_N. Let $\{K_\alpha\}$ be an admissible system satisfying (2.8), (2.9), (2.10), (2.11) and denote $I_\alpha=[a_\alpha,b_\alpha]=\operatorname{supp}\hat{K}_\alpha$. Thus $b_\alpha-a_\alpha\sim\min(a_\alpha^{1/2},(N-b_\alpha)^{1/2})$. By the previous discussion, $[e^{2\pi ik\theta}\mid k\in I_\alpha]$ is contained in a space S_α, $S_\alpha\subset[e^{2\pi ik\theta}\mid k\in\tilde{I}_\alpha]$ and $d(S_\alpha,l^\infty_{\dim S_\alpha})<c$. Here $\tilde{I}_\alpha$ is the interval with the same center as I_α, $|\tilde{I}_\alpha|=2|I_\alpha|$ and the norm on S_α is the uniform norm (clearly equivalent to the $\|\ \|_A$-norm). Denote "$\bigoplus$" the direct sum in l^∞-sense, $i_\alpha:[e^{2\pi ik\theta}\mid k\in I_\alpha]\to S_\alpha$ and $j_\alpha:S_\alpha\to X$ the formal inclusion maps. Since, by (2.9), $f=\Sigma_\alpha(f*K_\alpha)$ for $f\in X$, the maps $T_1;X\to\bigoplus_\alpha S_\alpha$, $T_2:S_\alpha\to X$ given by

$$T_1f=(f*K_\alpha)\quad\text{and}\quad T_2(\{g_\alpha\})=\sum g_\alpha$$

have a composition $T_2T_1=\mathrm{Id}_X$. The boundedness of T_1 is obvious while the boundedness of T_2 follows from the fact that if K is admissible, then supp $\hat{K}$ only intersects a bounded number of intervals $\tilde{I}_\alpha$ (hence K acts only on a bounded number of g_α's). This factorization proves the P_λ-property of X.

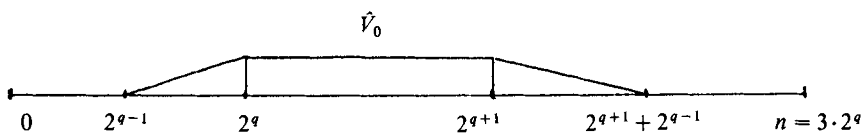
REMARK. The dimension of the l^∞-space $\bigoplus_\alpha S_\alpha$ containing X is of the order of N. In fact, one may obtain embedding in $l^\infty_{(1+\varepsilon)N}$, for any $\varepsilon > 0$. This follows from the previous argument and has been observed earlier by B. Kashin. We will use the previous procedure later, applied to certain (P_λ)-subspaces Y of X, and it will be of importance to get the l^∞-superspace of dimension proportional to dim Y. This property will be immediate from the description of these subspaces.

3. Construction of a polynomial basis

Consider the space $[1, e^{2\pi i\theta}, \dots, e^{2\pi in\theta}]$ and assume n of the form $3 \cdot 2^q$ (this hypothesis will not be a problem later on).

Let V_0 be the polynomial with trapezoïdal Fourier transform

$$\hat{V}_0(2^q) = 1$$



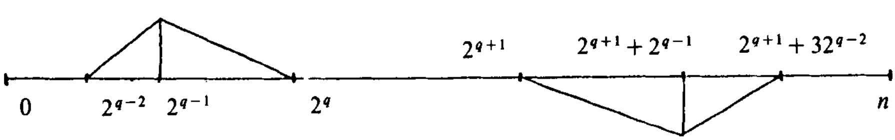
and denote B_0 the linear span of the functions $V_0(\theta - j/32^{q-1})$, where $0 \leqq j < 32^{q-1}$. Clearly, as in the previous section, these functions form an l^∞-basis (up to normalization). Moreover

$$\begin{aligned} B_0 = [&e^{2\pi ik\theta}(2^q < k < 2^{q+1}); \\ (3.1) \qquad &(k2^{-q+1} - 1)e^{2\pi ik\theta} + (2 - k2^{-q+1})e^{2\pi i(k+32^{q-1})}(2^{q-1} \leqq k \leqq 2^q)]. \end{aligned}$$

Next, consider V_1 with Fourier transform

$$\hat{V}_1(2^{q-1}) = 1$$

$$\hat{V}_1(n - 2^{q-1}) = -1$$



and denote B_1 the linear span of the functions $V_1(\theta - j/3\cdot 2^{q-2})$ where $0 \leqq j < 3\cdot 2^{q-2}$. If W_1 is the function with Fourier-transform

one clearly has

$$\sup_\theta \left| \sum a_j V_1\left(\theta - \frac{j}{3\cdot 2^{q-2}}\right)\right| \sim \sup_\theta \left| \sum a_j W_1\left(\theta - \frac{j}{3\cdot 2^{q-2}}\right)\right| . \tag{3.2}$$

The advantage of W_1 on V_1 is that now for $0 \leqq j, j' < 3\cdot 2^{q-2}$,

$$W_1\left(\frac{j'-j}{3\cdot 2^{q-2}}\right) = 3\cdot 2^{q-2}\delta_{j,j'} \tag{3.3}$$

showing that the functions $W_1(\theta - j/3\cdot 2^{q-2})$, hence $V_1(\theta - j/3\cdot 2^{q-2})$ $(0 \leqq j < 3\cdot 2^{q-2})$ are equivalent with the l^∞-basis, up to normalization (the upper estimate is obtained as in (2.14)). Thus again $B_1 \simeq l^\infty_{3\cdot 2^{q-2}}$. To see (3.3), write

$$\mathrm{W}_1(\theta) = \sum_{2^{q-2} < k \leqq 2^{q-1}} (k\cdot 2^{-q+2} - 1)e^{2\pi i k\theta} \tag{3.4}$$

$$+ \sum_{2^{q-2} < k < 2^q} (2 - k\cdot 2^{-q+1})e^{2\pi i k\theta} \tag{3.5}$$

$$+ \sum_{2^{q+1} < k < 2^{q+1} + 2^{q-1}} (k\cdot 2^{-q+1} - 4)e^{2\pi i k\theta} \tag{3.6}$$

$$+ \sum_{2^{q+1} + 2^{q-1} \leqq k < 2^{q+1} + 3\cdot 2^{q-2}} (11 - k\cdot 2^{-q+2})e^{2\pi i k\theta}. \tag{3.7}$$

Evaluate at $\theta = (j' - j)/3\cdot 2^{q-2}$ and replace in (3.6) k by $k - 3\cdot 2^{q-1}$, in (3.7) k by $k - 3\cdot 2^{q-1} - 3\cdot 2^{q-2}$. One gets

$$\sum_{2^{q-2} < k \leqq 2^{q-1}} (k\cdot 2^{-q+2} - 1)e^{2\pi i k\theta} + \sum_{2^{q-1} < k \leqq 2^q} (2 - k\cdot 2^{-q+1})e^{2\pi i k\theta}$$

$$+ \sum_{2^{q-1} < k < 2^q} (k\cdot 2^{-q+1} - 1)e^{2\pi i k\theta} + \sum_{2^{q-2} \leqq k < 2^{q-1}} (2 - k\cdot 2^{-q+2})e^{2\pi i k\theta}$$

$$= \sum_{2^{q-2} \leqq k < 2^q} e^{2\pi i k\theta}$$

clearly implying (3.3).

Further, it is easily checked that

$$B_1 = [(2 - k\cdot 2^{-q+1})e^{2\pi ik\theta} - (k\cdot 2^{-q+1} - 1)e^{2\pi i(k+3\cdot 2^{q-1})\theta}(2^{q-1} < k < 2^q);$$
(3.8)
$$(k\cdot 2^{-q+2} - 1)e^{2\pi ik\theta} - (2 - k\cdot 2^{-q+2})e^{2\pi i(k+9\cdot 2^{q-2})\theta}(2^{q-2} \leqq k \leqq 2^{q-1})].$$

Hence

(3.9) $$B_1 \perp B_0$$

and

$$B_0 + B_1 = [e^{2\pi ik\theta}(2^{q-1} < k < n - 2^{q-1});$$
(3.10)
$$(k\cdot 2^{-q+2} - 1)e^{2\pi ik\theta} - (2 - k\cdot 2^{-q+2})e^{2\pi i(k+9\cdot 2^{q-2})\theta}(2^{q-2} \leqq k \leqq 2^{q-1})].$$

To introduce the space B_2, define V_2 with Fourier transform

and let B_2 be the space generated by the translates $V_2(\theta - j/3\cdot 2^{q-3})$, $0 \leqq j < 3\cdot 2^{q-3}$. These translations satisfy

(3.11) $$V_2\left(\frac{j' - j}{3\cdot 2^{q-3}}\right) = 3\cdot 2^{q-3}\delta_{jj'}$$

and hence form, up to normalization, an l^∞-basis. Also

$$B_2 = [(2 - k\cdot 2^{-q+2})e^{2\pi ik\theta} + (k\cdot 2^{-q+2} - 1)e^{2\pi i(k+9\cdot 2^{q-2})}(2^{q-2} < k < 2^{q-1});$$
(3.12)
$$(k\cdot 2^{-q+3} - 1)e^{2\pi ik\theta} + (2 - k\cdot 2^{-q+3})e^{2\pi i(k+21\cdot 2^{q-3})}(2^{q-3} \leqq k \leqq 2^{q-2})].$$

Hence $B_2 \perp B_0, B_1$ and

$$B_0 + B_1 + B_2 = [e^{2\pi ik\theta}(2^{q-2} < k < n - 2^{q-2});$$
(3.13)
$$(k\cdot 2^{-q+3} - 1)e^{2\pi ik\theta} + (2 - k\cdot 2^{-q+3})e^{2\pi i(k+21\cdot 2^{q-3})}(2^{q-3} \leqq k \leqq 2^{q-2})].$$

The continuation of this process is by now clear. According to the parity of s, the function V_s satisfies

$$\hat{V}_s(2^{q-s}) = 1 = \hat{V}_s(n - 2^{q-s}) \qquad \text{for } s \text{ even},$$

$$\hat{V}_s(2^{q-s}) = 1 = -\hat{V}_s(n - 2^{q-s}) \qquad \text{for } s \text{ odd}.$$

Further

$$[1, e^{2\pi i\theta}, \dots, e^{2\pi i n\theta}] = B_1 + B_2 + \cdots + B_{q-1} + B_q \tag{3.14}$$

where

$$d(B_s, l^\infty_{\dim B_s}) < c \tag{3.15}$$

(B_q is generated by 1, $e^{2\pi i\theta}$, $e^{2\pi i(n-1)\theta}$, $e^{2\pi i n\theta}$). Also

$$\begin{aligned} B_0 + \cdots + B_s & \\ &= [e^{2\pi ik\theta}(2^{q-s} < k < n - 2^{q-s}); \\ &\quad (k \cdot 2^{-q+s+1} - 1)e^{2\pi ik\theta} \\ &\qquad + (-1)^s(2 - k\cdot 2^{-q+s+1})e^{2\pi i(k+n-32^{q-s-1})\theta}(2^{q-s-1} \leqq k \leqq 2^{q-s})] \end{aligned} \tag{3.16}$$

and the Lebesgue function $K_s(\theta, \psi)$ corresponding to the orthogonal projection on $B_0 + \cdots + B_s$ equals

$$\begin{aligned} K_s(\theta, \psi) &= \Lambda_s(\theta - \psi) \\ &\qquad + \Gamma_s(\theta - \psi)[e^{2\pi i(n-3\cdot 2^{q-s-1})\theta} + e^{-2\pi i(n-3\cdot 2^{q-s-1})\psi}] \end{aligned} \tag{3.17}$$

where

$$\hat{\Lambda}_s(k) \begin{cases} (k\cdot 2^{-q+s+1} - 1)^2[(k\cdot 2^{-q+s+1} - 1)^2 \\ \qquad\qquad + (2 - k\cdot 2^{-q+s+1})^2]^{-1}, & 2^{q-s-1} \leqq k \leqq 2^{q-s}, \\ 1, & 2^{q-s} < k < n - 2^{q-s}, \\ ((n-k)2^{-q+s+1} - 1)^2 \\ \times [((n-k)2^{-q+s+1} - 1)^2 \\ \qquad + (2 - (n-k)2^{-q+s+1})^2]^{-1}, & n - 2^{q-s} \leqq k \leqq n - 2^{q-s}, \\ 0, & \text{elsewhere}, \end{cases} \tag{3.18}$$

and

$$\hat{\Gamma}_s(k) = \begin{cases} (-1)^s(k\cdot 2^{-q+s+1}-1)(2-k\cdot 2^{-q+s+1}) & \\ \times[(k\cdot 2^{-q+s+1}-1)^2+(2-k\cdot 2^{-q+s+1})^2]^{-1} & \text{if } 2^{q-s-1}<k<2^{q-s}, \\ 0 & \text{elsewhere.} \end{cases}$$

(3.19)

The reader will easily verify that

$$\| \Gamma_s \|_1 < c, \tag{3.20}$$

and $\hat{\Lambda}_s(k) = 1 - \hat{\Lambda}'_s(k)$ for $0 \leqq k \leqq n$, Λ'_s satisfying also

$$\| \Lambda'_s \|_1 < c. \tag{3.21}$$

It follows from this discussion that, for $f \in [1, \ldots, e^{2\pi i n\theta}]$,

$$\left| \int_0^1 f(\psi) K_s(\theta, \psi) d\psi \right| \leqq c \| f \|_\infty. \tag{3.22}$$

The union of the l^∞-basis in the consecutive B_s-spaces thus yields a basis for $A_{n+1} = [1, \ldots, e^{2\pi i n\theta}]_\infty$, with bounded basis constant.

4. Finite decompositions

One of the ingredients in proving the isomorphism is the following abstract lemma:

LEMMA 4.1. *Let X be a finite dimensional normed space and assume*

$$X = X_0 \supset X_1 \supset X_2 \supset \cdots \supset X_{t-1} \supset X_t = \{0\} \tag{4.1}$$

a decreasing sequence of subspaces of X, such that the following conditions are fulfilled:

$$\dim X_s = d_s \qquad \text{with } d_s \text{ exponentially decreasing}, \tag{4.2}$$

$$X_s \simeq X_{s+1} \oplus l^\infty_{d_s - d_{s+1}}, \tag{4.3}$$

$$l^\infty_{N_s} \simeq X_s \oplus C_s \qquad \text{with } N_s \simeq d_s. \tag{4.4}$$

Then $d(X, l^\infty_{d_0})$ is bounded.

COMMENTS. The symbol $\simeq$ refers to isomorphism up to some bound. The final estimate on $d(X, l^\infty_{d_0})$ is only dependent on this bound. Condition (4.4) means that X_s is a P_λ-space, $\lambda < C$, and may be embedded in l^∞ of proportional dimension.

Proof of (4.1). Fix a sufficiently large index s_0 which will be specified later. Write, denoting again by $\oplus$ direct sums in l^∞ sense,

$$\begin{aligned}
&(l^\infty_{N_{s_0}} \oplus l^\infty_{d_{s_0}-d_{s_0+1}}) \oplus (l^\infty_{N_{s_0+1}} \oplus l^\infty_{d_{s_0+1}-d_{s_0+2}}) \oplus \cdots \oplus (l^\infty_{N_{t-1}} \oplus l^\infty_{d_{t-1}}) \\
&\quad \simeq (X_{s_0} \oplus C_{s_0} \oplus l^\infty_{d_{s_0}-d_{s_0+1}}) \oplus (X_{s_0+1} \oplus C_{s_0+1} \oplus l^\infty_{d_{s_0+1}-d_{s_0+2}}) \\
&\qquad \oplus \cdots \oplus (X_{t-1} \oplus C_{t-1} \oplus l^\infty_{d_{t-1}}) \qquad \text{(using 4.4)} \\
&\quad \simeq X_{s_0} \oplus (C_{s_0} \oplus l^\infty_{d_{s_0}-d_{s_0+1}} \oplus X_{s_0+1}) \oplus (C_{s_0+1} \oplus l^\infty_{d_{s_0+1}-d_{s_0+2}} \oplus X_{s_0+2}) \\
&\qquad \oplus \cdots \oplus (C_{t-1} \oplus l^\infty_{d_{t-1}}) \\
&\quad \simeq X_{s_0} \oplus (C_{s_0} \oplus X_{s_0}) \oplus (C_{s_0+1} \oplus X_{s_0+1}) \oplus \cdots \oplus (C_{t-1} \oplus X_{t-1}) \qquad \text{(using 4.3)} \\
&\quad \simeq X_{s_0} \oplus l^\infty_{N_{s_0}} \oplus l^\infty_{N_{s_0+1}} \oplus \cdots \oplus l^\infty_{N_{t-1}} \qquad \text{(using 4.4 again).}
\end{aligned}$$

Hence, the conclusion is that

$$(4.5) \qquad l^\infty_{N_{s_0}+N_{s_0+1}+\cdots+N_{t-1}+d_{s_0}} \simeq X_{s_0} \oplus l^\infty_{N_{s_0}+\cdots+N_{t-1}}.$$

Iterating (4.3), also

$$(4.6) \qquad \begin{aligned}
X &\simeq X_1 \oplus l^\infty_{d_0-d_1} \\
&\simeq X_2 \oplus l^\infty_{d_0-d_2} \\
&\ \ \vdots \\
&\simeq X_{s_0} \oplus l^\infty_{d_0-d_{s_0}}
\end{aligned}$$

(the isomorphism constant depends on s_0, which anyway is a bounded number depending on the exponential decay of d_s and the ratio N_s/d_s). Choosing indeed s_0 such that

$$(4.7) \qquad d_0 - d_{s_0} \geqq N_{s_0} + \cdots + N_{t-1},$$

then, by (4.5) and (4.6),

$$X \simeq l^\infty_{N_{s_0}+\cdots+N_{t-1}+d_{s_0}} \oplus l^\infty_{d_0-d_{s_0}-N_{s_0}-\cdots-N_{t-1}} = l^\infty_{d_0}$$

proving the lemma.

5. Construction of subspaces

Consider again the space $X = [1, e^{2\pi i\theta}, \ldots, e^{2\pi iN\theta}]_A$. Our purpose is to introduce a sequence of subspaces of X satisfying the conditions of Lemma 4.1.

Consider a partition of the interval $[0, N]$ in disjoint consecutive intervals $I_\alpha = [a_\alpha, b_\alpha]$ such that

$$b_\alpha - a_\alpha = 3 \cdot 2^{q_\alpha} \sim \min(a_\alpha^{1/2}, (N - b_\alpha)^{1/2}). \tag{5.1}$$

Fix α and identify $[e^{2\pi i k\theta} \mid a_\alpha \leqq k \leqq b_\alpha]$ with $A_{|I_\alpha|}$. The construction of Section 3 gives a decomposition

$$[e^{2\pi i k\theta} \mid k \in I_\alpha] = B_1^\alpha + B_2^\alpha + \cdots + B_{q_\alpha}^\alpha. \tag{5.2}$$

Define the following subspaces of X:

$$\begin{aligned} X_1 &= [B_r^\alpha \mid 2 \leqq r \leqq q_\alpha, \alpha \text{ arbitrary}], \\ X_2 &= [B_r^\alpha \mid 3 \leqq r \leqq q_\alpha, \alpha \text{ arbitrary}], \\ &\vdots \\ X_s &= [B_r^\alpha \mid s+1 \leqq r \leqq q_\alpha, \alpha \text{ arbitrary}], \\ &\vdots \end{aligned} \tag{5.3}$$

of exponentially decreasing dimension and terminating with the null space. (If $q_\alpha \leqq s$, then $[e^{2\pi i k\theta} \mid k \in I_\alpha]$ does not contribute to X_s.) Define P_s as the orthogonal projection from X onto X_s. Thus

$$P_s = \sum_{\alpha; q_\alpha > s} (R_\alpha - Q_{\alpha,s}) \tag{5.4}$$

where

$$R_\alpha f = \sum_{k \in I_\alpha} \hat{f}(k) e^{2\pi i k\theta}, \tag{5.5}$$

$$Q_{\alpha,s} = \text{orthogonal projection on } B_1^\alpha + \cdots + B_s^\alpha \text{ in } [e^{2\pi i k\theta} \mid k \in I_\alpha]. \tag{5.6}$$

The $Q_{\alpha,s}$-projections have been described in detail in Section 3.

6. End of the proof

We verify the conditions of Lemma 4.1.

Let $\Lambda_{\alpha,s}$ (resp. $\Gamma_{\alpha,s}$) be given by (3.18) (resp. 3.19), for $q = q_\alpha$. Then

$$P_s f(\theta) =$$

$$\sum_{\alpha; q_\alpha > s} \sum_{k \in I_\alpha} (1 - \hat{\Lambda}_{\alpha,s}(k - a_\alpha)) \hat{f}(k) e^{2\pi i k\theta} \tag{6.1}$$

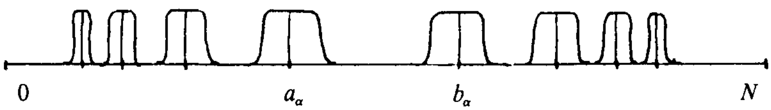
$$- \sum_{\alpha; q_\alpha > s} \sum_k \hat{\Gamma}_{\alpha,s}(k - a_\alpha) \hat{f}(k) e^{2\pi i (k + 3 \cdot 2^{q_\alpha}(1 - 2^{-s-1}))\theta} \tag{6.2}$$

$$- \sum_{\alpha; q_\alpha > s} \sum_k \hat{\Gamma}_{\alpha,s}(k - b_\alpha + 3 \cdot 2^{q_\alpha - s - 1}) \hat{f}(k) e^{2\pi i (k - 3 \cdot 2^{q_\alpha}(1 - 2^{-s-1}))\theta}. \tag{6.3}$$

By (3.20), the individual α-terms in (6.2) and (6.3) are given by bounded operators on $L^\infty(\Pi)$ and on X. By construction of the partition $\{I_\alpha\}$ of $[0, N]$, admissible kernels only act on a bounded number of terms. Hence (6.2) and (6.3) give bounded operators on X.

Consider now (6.1). The individual α-terms are not well-bounded operators on $L^\infty(\Pi)$ but their sum is a bounded operator on X. The operator is indeed given by a multiplier with following shape:

(6.4)



as the reader will easily verify from (3.18). Observe that the multiplier is supported by $2^{-s}|I_\alpha|$-size neighborhoods of the endpoints of the I_α-intervals. Admissible kernels only interfere with a bounded number of "clocks", each of them defining a bounded operator on $L^\infty(\Pi)$ and on X. Hence (6.1) also is bounded on X.

It follows from this discussion that P_s is bounded for the $\| \ \|_A$-norm on X and hence the X_s are $\mathcal{P}_\lambda$-spaces for some bounded λ. It also follows from the definition of X_s that

$$(6.5) \qquad \begin{aligned} X_s \subset [e^{2\pi i k\theta} \mid\, & a_\alpha \leqq k \leqq a_\alpha + 2^{-s}|I_\alpha| \text{ or} \\ & b_\alpha - 2^{-s}|I_\alpha| \leqq k \leqq b_\alpha, \text{ for some } \alpha]. \end{aligned}$$

The procedure discussed at the end of Section 2 therefore permits one to embed X_s in $l^\infty_{N_s}$, where $N_s \sim 2^{-s}N \sim \dim X_s$.

At this point, conditions (4.2) and (4.4) are checked.

It remains to identify the orthogonal complement of X_{s+1} in X_s. By construction

$$(6.6) \qquad X_s = X_{s+1} + \Big(\mathop{+}\limits_{\alpha; q_\alpha > s} B^\alpha_{s+1} \Big).$$

The individual B^α_{s+1}-spaces are uniformly isomorphic to l^∞ for the L^∞ and $\| \ \|_A$-norm. Again admissible kernels only interfere with a bounded number of these spaces. Hence, if $\xi_\alpha \in B^\alpha_{s+1}$,

$$(6.7) \qquad \Big\| \sum \xi_\alpha \Big\|_A \leqq c \max_\alpha \| \xi_\alpha \|_\infty .$$

It remains to show the converse inequality. This is very easy, remembering the construction of the B_s spaces in Section 3. It follows that

(6.8) $B^{\alpha}_{s+1} \subset [e^{2\pi ik\theta} \mid 2^{q_\alpha - s-2} \leqq k - a_\alpha \leqq 2^{q_\alpha - s}; 2^{q_\alpha - s - 2} \leqq b_\alpha - k \leqq 2^{q_\alpha - s}]$.

Hence, for given α, there is a function G_α such that

(6.9) G_α is a sum of 2 admissible kernels,

(6.10) $\xi * G_\alpha = \xi \quad \text{for } \xi \in B^{\alpha}_{s+1}$,

(6.11) $\xi * G_\alpha = 0 \quad \text{for } \xi \in B^{\alpha'}_{s+1}, \quad \alpha' \neq \alpha$.

(6.11) just follows by letting supp $\hat{G}_\alpha \subset I_\alpha$.

Hence (6.7) is an equivalence and $\bigoplus_\alpha B^{\alpha}_{s+1}$ is an l^∞-space. Thus (4.3) holds and the hypothesis of the lemma is fulfilled. This concludes the proof of (1.3).

Appendix. Existence of a basis in the ball algebra

The purpose of what follows is to give an affirmative answer to the question considered in [Pel] (p. 68) on the existence of a basis in $A(B_d)$, the algebra of bounded analytic functions on B_d with continuous boundary values. For simplicity, we consider the case $d = 2$, but the argument easily generalizes. It is an existence proof, rather than an explicit construction. The method is based on simple Banach space techniques and some ideas to built projections, inspired by Section 3. We also use Wojtaszczyk's result that $A \equiv A(B_d)$ is isomorphic to its c_0-direct sum, hence

(1) $$A \cong A \oplus c_0$$

(see [Woj]).

Only few (and easy) facts from function theory will be of relevance here.

LEMMA 2. *Let A be a Banach space satisfying* (1) *and having an increasing sequence B_j of well-complemented finite dimensional subspaces, generating A, such that the projections commute and*

(3) $$B_{j+1} = B_j \oplus C_j$$

where the C_j are P_λ-spaces for some fixed λ (i.e. A has an FDD *consisting of P_λ-spaces). Then A has a basis.*

PROOF. Since the C_j are P_λ-spaces for bounded λ, one may write

(4) $$l^\infty_{N_j} = C_j \oplus D_j$$

for some increasing sequence of integers N_j (such that C_j may be imbedded in the corresponding l^∞-space). Thus

(5) $$E_{j+1} \equiv B_{j+1} \oplus \bigoplus_{k=1}^{j} D_k$$

is an increasing sequence of well-complemented subspaces of $E \equiv A \oplus \bigoplus_{k=1}^{\infty} D_k$, with commuting projections generating E, and with difference spaces the spaces $l_{N_j}^\infty$, i.e. by (3), (4)

(6) $$E_{j+1} = E_j \oplus C_j \oplus D_j = E_j \oplus l_{N_j}^\infty .$$

In the definition of E, the direct sum of the D_k-spaces is taken in c_0-sense. Hence

(7) $$\bigoplus_{k=1}^{\infty} D_k \cong c_0.$$

It is obvious from the preceding that the space E has a basis. Since, by (1), $A \cong A \oplus c_0 \cong E$, so does A.

Denote again $W_n = [z^j w^{n-j} \mid 0 \leqq j \leqq n]$ the spaces of homogeneous polynomials and P_n the orthogonal projection on W_n, given by the formula (cf. 1.2)

(8) $$P_n f = \frac{\int f(\eta) \langle \zeta, \eta \rangle^n \sigma(d\eta)}{\int |\eta_1|^{2n} \sigma(d\eta)} .$$

LEMMA 9. (1) *If* (λ_n) *is a bounded multiplier on the disc-algebra, i.e.*

(10) $$\sup_{0 \leqq \theta \leqq 1} \left| \sum_0^\infty \lambda_n a_n e^{2\pi i n \theta} \right| \leqq C \sup_{0 \leqq \theta \leqq 1} \left| \sum_0^\infty a_n e^{2\pi i n \theta} \right| ,$$

then $\Sigma \lambda_n P_n$ *defines a bounded operator on* $A(B_2)$.

(2) *Assume* $\lambda = (\lambda_n)$ *fulfils* (10) *and* $\lambda_n = 0$ *outside the interval* $[N, 2N]$ *(*N *is arbitrary). Then* $\Sigma \lambda_n P_n$ *acts boundedly on* $L^\infty(S_2)$.

PROOF. (1) is well-known (see [Ru]) and follows from a transference argument. To get (2), one has to verify that

(11) $$\int_{S_2} \left| \sum_N^{2N} \lambda_n (n+1) \eta_1^n \right| \sigma(d\eta) < C'(C),$$

hence

$$\int\int\left|\sum_N^{2N}\lambda_n(n+1)\rho^{n/2}e^{2\pi in\theta}\right|d\rho\,d\theta<C. \tag{12}$$

The left member of (12) is clearly bounded by

$$CN\int\rho^{N/2}\left\{\left\|\sum_0^N\rho^{m/2}\lambda_{m+N}e^{2\pi im\theta}\right\|_{L^1(d\theta)}\right\}d\rho$$

$$\leqq C\cdot\sup_{0<\rho<1}\left\|\sum_{-\infty}^{\infty}\rho^{|m|/2}e^{2\pi im\theta}\right\|_{L^1(d\theta)}<C$$

since $(\lambda_{m+N})_{m\in\mathbf{Z}}$ is by hypothesis also a bounded multiplier on $L^1(\pi)$.

The next fact belongs to geometry of Banach spaces (see [T-J] for more details on these matters based on theory of absolutely summing operators).

Lemma 13. *Let X be an n-dimensional subspace of L^∞, $i: X\to L^\infty$ the inclusion map and $T: L^\infty\to X$ a bounded linear operator, i.e. $\|T\|<C$. If* Trace$(Ti)\sim n$, *i.e.* Trace$(Ti)>cn$ *where $c>0$ is thought of as a given constant, then X contains an l_m^∞-isomorph for $m\sim n$, i.e. there are vectors $e_1,\dots,e_m$ in X satisfying*

$$C^{-1}\max|a_k|\leqq\left\|\sum_1^m a_ke_k\right\|\leqq C\max|a_k|. \tag{14}$$

Considering $W_N+W_{N+1}+\cdots+W_{2N}=X$ as subspace of $L^\infty(S_2)$, thus

$$\dim X\sim N^2, \tag{15}$$

and applying (9.2), taking for $(\lambda_n)_{N\leqq n\leqq 2N}$ a multiplier with usual trapezoïdal shape, yields the following consequence of Lemma 13:

Lemma 16. *The spaces $W_N+\cdots+W_{2N}$, considered as subspaces of $A(B_2)$, contain l^∞-isomorphs of dimension $\sim N^2$.*

The following lemma is a simple and straightforward approximation of Poisson integrals by averages over a partition of S_2 constructed from suitable-size balls for the non-isotropic metric (cf. [Ru]). The details of the argument are left to the reader.

Lemma 17. *For each space $W_0+W_1+\cdots+W_N$ there is a subspace Y of $L^\infty(S_2)$,* $\dim Y=D\sim N^2$, *Y isometric to l_D^∞ and such that any function f in $W_0+\cdots+W_N$ of $\|f\|_\infty\leqq 1$ has an approximation by some $\varphi\in Y$ in the sense that*

(18) $$\| f - \varphi \|_{L^\infty(S_2)} < 10^{-1}.$$

LEMMA 19. *In the situation of Lemma 17, one may get an l_D^∞-isomorph Z contained in $L^\infty(S_2)$ and actually containing $W = W_0 + W_1 + \cdots + W_N$.*

PROOF. The restriction j to W of a projection p, $\| p \| = 1$, from $L^\infty(S)$ onto Y is an isomorphism onto a subspace W' of Y. Let $u : Y \to L^\infty(S)$ be an extension of the inverse $j^{-1} : W' \to W$. Define Z as image of $u + T(\mathrm{Id} - pu)$, where T is a suitable isomorphism from $L^\infty(S)$ into itself.

We now formulate the main lemma:

LEMMA 20. *For each N, there is a subspace B_N of $W_0 + \cdots + W_{C\cdot N}$ containing $W_0 + \cdots + W_N$ and uniformly complemented in $A = A(B_2)$ by a projection Q satisfying $W_0 + \cdots + W_N \subset \mathrm{Ker}(I - Q^*)$. Here C is some integer constant.*

From this, the proof on the existence of a basis in A is easily concluded. Let $\lambda^{(N)}$ be piecewise linear on $\mathbf{Z}_+$ such that

$$\begin{cases} \lambda_n^{(N)} = 1 & \text{if } n \leqq C \cdot N, \\ \lambda_n^{(N)} = 0 & \text{if } n > 2C \cdot N. \end{cases}$$

Let Q_N be defined by $\bar{Q}_N = Q_{N^\circ}(\Sigma\, \lambda_n^{(N)} P_n)$. Then $\bar{Q}_N$ is still a bounded projection on B_N and the difference operators $Q_{C\cdot N} - Q_N$ are bounded on $L^\infty(S)$, as the reader will easily check, invoking Lemma 9.2. Thus the spaces $B_j' = B_{C^j}$ have difference spaces which are P_λ, for some constant λ, and are complemented by commuting projections. It remains to apply Lemma 2.

We first reduce Lemma 20 to proving

LEMMA 21. *For each N, there is a subspace V_N of the direct sum $A \oplus l^\infty_{C\cdot N^2}$, V_N containing $W_0 + \cdots + W_N$ and contained in $(W_0 + \cdots + W_{2N}) \oplus l^\infty_{C\cdot N^2}$ and V_N well-complemented in $A \oplus l^\infty_{C\cdot N^2}$ by a projection Γ such that $W_0 + \cdots + W_N \subset \mathrm{Ker}(I - \Gamma^*)$. Here C stands again for some numerical constant.*

Lemma 20 may then be derived from Lemmas 16 and 21. Let μ be piecewise linear on $\mathbf{Z}_+$.

$$\begin{cases} \mu_n = 1 & \text{if } n \leqq 2N, \\ \mu_n = 0 & \text{if } n > 3N. \end{cases}$$

Choose an integer $K > 4$ and sufficiently large to ensure that $W_{KN} + W_{KN+1} + \cdots + W_{2KN}$ contains an l^∞-isomorph L of dimension $= C \cdot N^2$, where C is the constant appearing in Lemma 2. Let ν be piecewise linear on $\mathbf{Z}_+$, such that

$$\begin{cases} \nu_n = 1 & \text{if } KN \leqq n \leqq 2KN, \\ \nu_n = 0 & \text{if } n < (K-1)N \text{ or } n > (2K+1)N. \end{cases}$$

Thus μ and ν are disjointly supported. Denote p a projection from A onto L. Clearly, if $u : l^\infty_{CN^2} \to L$ is an isomorphism and π_i $(i = 1, 2)$ the projections from $A \oplus l^\infty_{C \cdot N^2}$ onto the first and second component, the operator

$$Q = (\pi_1 + u\pi_2) \circ \Gamma \circ ((\Sigma\, \mu_n P_n) + u^{-1} p(\Sigma\, \nu_n P_n)) \tag{22}$$

defines a (well)-bounded projection from A onto $(\pi_1 + u\pi_2)(V_N) = B_N$. The space B_N and projection Q are easily seen to satisfy the conditions of Lemma 20. The constant C in Lemma 20 may be taken as $2K$.

In proving Lemma 21, take N of the form $N = 2^j$. We first introduce a subspace M of

$$\begin{aligned} &A \oplus (W_0 + \cdots + W_{2N}) \oplus (W_0 + \cdots + W_N) \\ &\oplus (W_0 + \cdots + W_{N/2}) \oplus \cdots \oplus W_0 \end{aligned} \tag{23}$$

considered as subspace of

$$A' \equiv A \oplus \underbrace{L^\infty(S_2) \oplus \cdots \oplus L^\infty(S_2)}_{(j+2)\text{ components}} \tag{24}$$

where $\oplus$ stands for direct sum in l^∞-sense. Moreover, the natural orthogonal projection from A' onto M will be well-bounded. The idea of the following construction has similarities with the procedure appearing in Section 3 of this paper. Denote

$$\begin{aligned} F_{2N} &= W_0 + \cdots + W_{2N}, \\ F_N &= W_0 + \cdots + W_N, \\ F_{N/2} &= W_0 + \cdots + W_{N/2}, \\ &\vdots \\ F_0 &= W_0 \end{aligned}$$

the different components appearing in (23). Denote further for $j + k \leqq l$ by $e(j, k, l)$ the monomial $z^j w^k$ considered as element of F_l $(l = 0, 1, 2, 4, \ldots, 2N)$.

When writing $z^j w^k$, we'll think of it as element of the first component A of (23). Let M be generated by the following elements:

$$\{z^j w^k \mid j+k=n, n>2N\}$$
$$\cup \left\{\left(\frac{n}{N}-1\right)^{1/2} z^j w^k + \left(2-\frac{n}{N}\right)^{1/2} e(j,k,2N) \,\Big|\, j+k=n, N<n \leqq 2N\right\}$$
$$\cup \left\{\left(\frac{2n}{N}-1\right)^{1/2} e(j,k,2N) + \left(2-\frac{2n}{N}\right)^{1/2} e(j,k,N) \,\Big|\, j+k=n, \frac{N}{2}<n \leqq N\right\}$$
$$\cup \left\{\left(\frac{4n}{N}-1\right)^{1/2} e(j,k,N) + \left(2-\frac{4n}{N}\right)^{1/2} e\left(j,k,\frac{N}{2}\right) \,\Big|\, j+k=n, \frac{N}{4}<n \leqq \frac{N}{2}\right\}$$
$$\cup \left\{\left(\frac{8n}{N}-1\right)^{1/2} e\left(j,k,\frac{N}{2}\right) + \left(2-\frac{8n}{N}\right)^{1/2} e\left(j,k,\frac{N}{4}\right) \,\Big|\, j+k=n, \frac{N}{8}<N \leqq \frac{N}{4}\right\}$$
$$\vdots$$

(25)

Denote Q the orthogonal projection onto M. Writing Q explicitly from (25) (similarly as in Section 3) and applying Lemma 9, one easily observes the boundedness of Q acting on A'.

In view of Lemma 19, each F_l is contained in an l^∞-isomorph Z_l, $\dim Z_l \sim 4^{-l}N^2$, Z_l a subspace of the corresponding $L^\infty(S_2)$-component in (24). Let Q_0 be the restriction of Q to the subspace

$$A'' = A \oplus Z_{2N} \oplus Z_N \oplus Z_{N/2} \oplus \cdots \oplus Z_0 \cong A \oplus l^\infty_{C\cdot N^2} \tag{26}$$

of A', containing M by construction. Define V as the orthogonal complement of M in A''. Thus $\Gamma = I - Q_0$ is the projection from A'' onto V and is bounded by the preceding. Since the monomials $z^j w^k$ are orthogonal on M for $j+k \leqq N$, V contains $W_0 + W_1 + \cdots + W_N$ as subspace of the A-component in (26). It is clear that V is contained in $(W_0 + \cdots + W_{2N}) \oplus Z_{2N} \oplus \cdots \oplus Z_0$. Finally, observe that $W_0 + \cdots + W_N$ as subspace of the A-component of A'' is contained in $\operatorname{Ker} Q_0^*$.

This concludes the proof of Lemma 21 and the existence of a basis in the ball algebra.

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